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ON THE GROUP OF CONFORMAL TRANSFORMATIONS OF A COMPACT RIEMANNIAN MANIFOLD. III

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1. Introduction

Let g_{ij}, R_{hijk}, $R_{ij} = R^k{}_{ijk}$ be respectively the metric, Riemann and Ricci tensors of a Riemannian manifold M^n of dimension n, and denote

$$(1.1) \qquad P = R^{hijk}R_{hijk}, \qquad Q = R^{ij}R_{ij} .$$

Throughout this paper all Latin indices take the values $1, \cdots, n$ unless stated otherwise, and repeated indices imply summation. In a recent paper [2] the author proved

Theorem 1. *Suppose that a compact Riemannian manifold $M^n (n > 2)$ with constant scalar curvature $R = g^{ij}R_{ij}$ admits an infinitesimal nonhomothetic conformal transformation v, and let L_v be the operator of the infinitesimal transformation v. If*

$$(1.2) \qquad a^2L_vP + b(2a + nb)L_vQ = const.,$$

where a and b are constants such that

$$(1.3) \qquad c \equiv 4a^2 + 2(n-2)ab + n(n-2)b^2 > 0 ,$$

then M^n is isometric to a sphere.

In particular, when $a = 0$ or $b = 0$, Theorem 1 is reduced to a result of Yano [4], which is a generalization of some results of Lichnerowicz [3] and the author [1]. Yano pointed out that condition (1.3) is equivalent to that a and b are not both zero.

Very recently, Yano and Sawaki [5] obtained the following theorem similar to Theorem 1:

Theorem 2. *Suppose that a compact Riemannian manifold M^n $(n > 2)$ with constant R admits an infinitesimal nonhomothetic conformal transformation v. If*

$$(1.4) \qquad L_vL_v[(n-2)a^2P + 4b(2a + b)Q] \leq 0 ,$$

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where a and b are constants such that $a + b \neq 0$, then M^n is isometric to a sphere.

The purpose of this paper is to generalize both Theorems 1 and 2 to

Theorem 3. *Suppose that a compact Riemannian manifold M^n ($n>2$) with constant R admits an infinitesimal nonhomothetic conformal transformation v. If*

$$L_v L_v\left(a^2P + \frac{c - 4a^2}{n-2}Q\right) \leq 0, \tag{1.5}$$

where

$$\begin{aligned} c \equiv 4a^2 + (n-2)[2a \sum_{i=1}^{4} b_i + (b_1 - b_2 + b_3 - b_4 + b_5 - b_6)^2 \\ - 2b_1b_3 - 2b_2b_4 + 2b_5b_6 + (n-1)\sum_{i=1}^{6} b_i^2] > 0, \end{aligned} \tag{1.6}$$

a and b's being constants, then M^n is isometric to a sphere.

It is obvious that Theorem 3 is reduced to a generalization of Theorem 1 when $b_2 = \cdots = b_6 = 0$, and to Theorem 2 when

$$b_1 = \cdots = b_4 = b/(n-2), \qquad b_5 = b_6 = 0.$$

We need the following theorem of Yano [5] to prove Theorem 3:

Theorem 4. *Suppose that a compact orientable Riemannian manifold M^n ($n > 2$) with constant R admits an infinitesimal nonhomothetic conformal transformation v so that*

$$L_v g_{ij} = 2\phi g_{ij}, \quad \phi = const., \tag{1.7}$$

and let ∇ denote the operator of covariant derivation of M^n with respect to g_{ij}. If

$$\int_{M^n} T_{ij}\phi^i\phi^j dA \geq 0, \tag{1.8}$$

where

$$T_{ij} = R_{ij} - \frac{R}{n} g_{ij}, \tag{1.9}$$

$\phi^i = \nabla^i\phi = g^{ij}\nabla_j\phi$, and dA is the element of area of M^n at a point, then M^n is isometric to a sphere.

2. Lemmas

Throughout this section M^n will always denote a compact orientable Riemannian manifold of dimension $n>2$. Let Δ be the Laplace-Beltrami operator on M^n. Then, for any scalar field f on M^n, (2.1) $$\Delta f = -\nabla^i\nabla_j f .$$

Thus we have

(2.2) $$\int_{M^n} \Delta f \, dA = 0$$

from the well-known Green's formula:

(2.3) $$\int_{M^n} \nabla^i \xi_i \, dA = 0 ,$$

where ξ_i is any vector field on M^n.

Lemma 1. *If a nonconstant scalar field ϕ on a manifold M^n satisfies $\Delta\phi = k\phi$, where k is constant, then k is positive.*

Proof. From equations (2.1), (2.2) we obtain

(2.4) $$\begin{aligned} 0 = \int_{M^n} \Delta(\phi^2) dA &= 2\int_{M^n} (\phi\Delta\phi - \phi^i\phi_i) dA \\ &= 2\int_{M^n} (k\phi^2 - \phi^i\phi_i) dA , \end{aligned}$$

which gives Lemma 1 immediately.

Lemma 2. *Let v be an infinitesimal conformal transformation on M^n so that*

(2.5) $$L_v g_{ij} = 2\phi g_{ij} .$$

Then

(2.6) $$\begin{aligned} L_v R_{hijk} = 2\phi R_{hijk} &- g_{hk}\nabla_j\phi_i + g_{hj}\nabla_k\phi_i \\ &- g_{ij}\nabla_k\phi_h + g_{ik}\nabla_j\phi_h , \end{aligned}$$

(2.7) $$L_v R_{ij} = g_{ij}\Delta\phi - (n-2)\nabla_j\phi_i ,$$

(2.8) $$L_v R = 2(n-1)\Delta\phi - 2R\phi .$$

Lemma 2 can be proved by a straightforward computation.

Lemma 3. *If M^n has constant R and admits an infinitesimal nonhomothetic conformal transformation v so that (1.7) holds, then*

(2.9) $$\Delta\phi = R\phi/(n-1) ,$$

(2.10) $$R > 0 .$$

Equation (2.9) follows from equation (2.8) due to the constancy of R, and equation (2.10) from Lemma 1.

Lemma 4 (*Yano and Sawaki* [5]). *If M^n admits an infinitesimal conformal transformation v so that equation (2.5) holds, then, for any scalar field f on M^n,*

$$\int_{M^n} \phi f \, dA = -\frac{1}{n} \int_{M^n} L_v f \, dA . \tag{2.11}$$

Proof. Substituting fv_i for ξ_i in the Green's formula (2.3) we obtain

$$\int_{M^n} f\nabla^i v_i \, dA = -\int_{M^n} v_i \nabla^i f \, dA = -\int_{M^n} L_v f \, dA . \tag{2.12}$$

On the other hand, since

$$L_v g_{ij} = \nabla_i v_j + \nabla_j v_i ,$$

from equation (2.5) we have $\nabla^i v_i = n\phi$, which and equation (2.12) yield the required equation (2.11) immediately.

3. Proof of Theorem 3

On the manifold M^n consider the covariant tensor field of order 4:

$$\begin{aligned} W_{hijk} = {} & aT_{hijk} + b_1 g_{hk} T_{ij} - b_2 g_{hj} T_{ik} + b_3 g_{ij} T_{hk} \\ & - b_4 g_{ik} T_{hj} + b_5 g_{hi} T_{jk} - b_6 g_{jk} T_{hi} , \end{aligned} \tag{3.1}$$

where

$$T_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{ij} g_{hk} - g_{ik} g_{hj}) , \tag{3.2}$$

and a and b's are constants satisfying (1.6). Then

$$W^{hijk} W_{hijk} = a^2 P + \frac{c - 4a^2}{n-2} Q - \frac{1}{n}\left(\frac{2a^2}{n-1} + \frac{c-4a^2}{n-2}\right) R^2 , \tag{3.3}$$

where c is defined by (1.6). From equation (3.3) it follows that

$$L_v(W^{hijk} W_{hijk}) = L_v\left((a^2 P + \frac{c - 4a^2}{n-2} Q\right) . \tag{3.4}$$

By assuming the infinitesimal nonhomothetic conformal transformation v to be defined by (1.7), from equations (3.1), (3.2), (1.9), (2.5), (2.6), (2.7), (2.9) we can easily obtain

$$
\begin{aligned}
L_v W_{hijk} = {} & 2a\phi R_{hijk} - [a + (n-2)b_1] g_{hk} \nabla_j \phi_i \\
& + [a + (n-2)b_2] g_{hj} \nabla_k \phi_i - [a + (n-2)b_3] g_{ij} \nabla_k \phi_h \\
& + [a + (n-2)b_4] g_{ik} \nabla_j \phi_h - (n-2) b_5 g_{hi} \nabla_k \phi_j + (n-2) b_6 g_{jk} \nabla_i \phi_h \\
& - \frac{\phi R}{n(n-1)} g_{ij} g_{hk} [4a + (3n-4)(b_1 + b_3)] \\
& + \frac{\phi R}{n(n-1)} g_{ik} g_{hj} [4a + (3n-4)(b_2 + b_4)] \\
& - \frac{3n-4}{n(n-1)} b_5 \phi R g_{hi} g_{jk} + \frac{3n-4}{n(n-1)} b_6 \phi R g_{jk} g_{hi} + 2b_1 \phi g_{hk} R_{ij} \\
& - 2b_2 \phi g_{hj} R_{ik} + 2b_3 \phi g_{ij} R_{hk} - 2b_4 \phi g_{ik} R_{hj} + 2b_5 \phi g_{hi} R_{jk} \\
& - 2b_6 \phi g_{jk} R_{ki} .
\end{aligned} \tag{3.5}
$$

Multiplying both sides of equation (3.5) by W^{hijk} and making use of equations (3.1), (3.2), (1.9), (3.3), (1.6) and $R^i{}_{ijk} = 0$, an elementary but lengthy calculation yields

$$W^{hijk} L_v W_{hijk} = 2\phi W^{hijk} W_{hijk} - c T^{ij} \nabla_j \phi_i . \tag{3.6}$$

By substituting equation (3.6) in the well-known formula

$$L_v (W^{hijk} W_{hijk}) + 2W^{hijk} L_v W_{hijk} - 8\phi W^{hijk} W_{hijk}, \tag{3.7}$$

we thus have

$$\phi L_v ({}^{hijk} W_{hijk}) = -4\phi^2 W^{hijk} W_{hijk} - 2c\phi T^{ij} \nabla_j \phi_i . \tag{3.8}$$

Since the manifold M^n is of constant R, it is known that fo $n < 2$

$$\nabla^j R_{ij} = 0 , \tag{3.9}$$

and therefore

$$\nabla^j T_{ij} = 0 . \tag{3.10}$$

Thus

$$\nabla^j (T_{ij} \phi \phi^i) = T_{ij} \phi^i \phi^j + \phi T_{ij} \nabla^j \phi^i . \tag{3.11}$$

Without loss of generality we may assume our manifold M^n to be orientable, since otherwise we need only to take an orientable twofold covering space of

M^n. Substituting equation (3.8) in equation (3.11), integrating the resulting equation over the manifold M^n and using equation (2.3) we obtain

$$2c\int_{M^n} T_{ij}\phi^i\phi^j dA = \int_{M^n} \phi L_v(W^{hijk}W_{hijk})dA + 4\int_{M^n} \phi^2 W^{hijk}W_{hijk}dA \,. \tag{3.12}$$

On the right side of equation (3.12), the second integral is nonnegative since its integrand is so, and the first integral is equal to, by Lemma 4 and equations (3.3), (1.5),

$$-\frac{1}{n}\int_{M^n} L_vL_v(W^{hijk}W_{hijk})dA = -\frac{1}{n}\int_{M^n} L_vL_v\left(a^2P + \frac{c-4a^2}{n-2}Q\right)dA \geq 0 \,.$$

Hence the integral on the left side of equation (3.12) is nonnegative, and Theorem 3 follows from Theorem 4 immediately.

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